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Some properties of approximate solutions for vector optimization problem with set-valued functions

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Abstract In this paper, we study the approximate solutions for vector optimization problem with set-valued functions. The scalar characterization is derived without imposing any convexity assumption on the objective functions. The relationships between approximate solutions and weak efficient solutions are discussed. In particular, we prove the connectedness of the set of approximate solutions under the condition that the objective functions are quasiconvex set-valued functions.

Keywords Vector optimization · Set-valued function · Scalarization · Approximate solution · Quasiconvex set-valued function

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1 Introduction

It is well known that optimization models describe only simplified versions of real problems, and numerical algorithms generate only approximate solutions. Moreover, the (weak) efficient solution set may be empty in the noncompact case, whereas approximate solutions always exist under very weak assumptions. Hence it is interesting and meaningful to have a theoretical analysis of the notion of approximate solution. The first and most popular concept was introduced by Kutateladze [10]. Loridan [13] introduced a notion of ε-efficient solutions

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for multiobjective programs (MOP), which was followed by White [21] who proposed several concepts of approximate solution for MOP. Since then, approximate solutions of MOP have been examined in the literature by many authors from different points of view. Existence conditions were developed by Deng [3] and Dutta and Vetriel [4] for convex MOP, while KKT-type conditions and saddle point conditions were derived by Dutta and Vetriel [4]. Connections between different definitions of approximate solutions were analyzed by Gutierrez [5], and so on. Tammer [18],[1] Tanaka [20], and others studied approximate solutions of vector optimization problems in general ordered vector spaces. Meanwhile, since optimization problems with set-valued objective functions are closely related to problems in stochastic programming, fuzzy programming, optimal control and the duality of vector optimization problems. Much attention has been paid to approximate solutions for vector optimization with set-valued functions. For example, Rong and Wu [17] introduced the notion of ε-weak efficient solution, and investigated scalar, Lagrangian multiplier, and duality properties for vector optimization problems with cone-subconvexlike set-valued functions based on the separation theorem of convex sets. Ling [11] gave scalar and Lagrangian multiplier results for ε-super efficient solution via generalized alternative theorem. It is worthwhile noticing that these results in [11,17] rely heavily on the separation theorems for convex sets. The first aim of this paper is to derive the scalar characterization without imposing any convexity assumption on objective functions. Since numerical algorithms generate only approximate solutions, the second aim is to discuss the relations between approximate solutions and weak efficient solutions, which generalize, improve and unify the corresponding results of Dutta and Vetrivel (see [4, Proposition 2.1]) and Deng (see [3, Proposition 1]). In particular, we prove the connectedness of the set of approximate solutions for vector optimization problems with set-valued functions under the assumption that the objective functions are quasiconvex set-valued functions, which generalize and improve the corresponding results of Helbig [6].

2 Preliminary

Throughout the paper, let X, Y be two Hausdorff locally convex topological vector spaces, Y^* be the topological dual space of Y and C be a closed convex pointed cone in Y with non-empty interior. A vector ordering in Y associated with the cone C is the relation $\leq$ defined by

$$x \leq y \Longleftrightarrow y - x \in C.$$

The dual cone of C is defined as

$$C^* = \{l \in Y^* : l(c) \geq 0, \quad \forall c \in C\}.$$

Let $M \subset Y$ be an arbitrary nonempty subset, the symbol $\overline{M}$, int M, and cone(M) denote the closure of M, the interior of M, the generated cone of M, respectively. Obviously, when M is a convex set, cone$(M) = \bigcup\{\lambda x : \lambda \geq 0, x \in M\}$.

A nonempty convex set $B \subset C$ is said to be *a base of C* if

(i) $0 \notin \overline{B}$;
(ii) $C = \text{cone}(B) = \bigcup\{\lambda x : \lambda \geq 0, x \in B\}$.

It is well known that a cone with a base must be pointed. Further discussions regarding cones can be found in [8,9].

[1] We note that Gerstwitz is actually the same person as Gerth C. and Tammer C.

Definition 2.1 [19]

(i) A function $\varphi : Y \to R$ is monotone if

$$y_1 - y_2 \in C \Rightarrow \varphi(y_1) \geq \varphi(y_2).$$

(ii) A function $\varphi : Y \to R$ is strictly monotone if

$$y_1 - y_2 \in \operatorname{int} C \Rightarrow \varphi(y_1) > \varphi(y_2).$$

Remark 2.1

(i) If φ is strictly monotone and continuous, then φ is monotone.
(ii) If $\varphi \in C^*\backslash\{0\}$, then φ is strictly monotone; $\varphi \in C^*$, then φ is monotone.

Lemma 2.1 [9] *Let C be a closed convex pointed cone with* $\operatorname{int} C \neq \emptyset$. *Then, for fixed* $e \in \operatorname{int} C$,

$$B^* = \{l \in C^* : l(e) = 1\}$$

is a w^*-compact base of C^*.

Remark 2.2

(1) Below, we list some examples of cones with nonempty interiors.

Example 2.1

(i) Let $Y = R^n, C_1 = \{x = (x_1, x_2, \ldots, x_n) \in R^n : x_i \geq 0, i = 1, 2, \ldots, n\}$. Then, $\operatorname{int} C_1 = \{x = (x_1, x_2, \ldots, x_n) \in R^n : x_i > 0, i = 1, 2, \ldots, n\}$.
(ii) Let $Y = C(G)$, where $G \subset R^n$ is a bounded closed set, let $C_2 = \{x \in C(G) : x(t) \geq 0, \forall t \in G\}$. Then, $\operatorname{int} C_2 = \{x \in C(G) : x(t) > 0, \forall t \in G\}$.
(iii) Let $Y = L^\infty(\Omega)$, where $\Omega \subset R^n, 0 < mes\Omega < \infty$, let $C_3 = \{x \in L^\infty(\Omega) : x(t) \geq 0, \forall \text{ a.e. } t \in \Omega\}$. Then, $\operatorname{int} C_3 \neq \emptyset$.
(iv) Let $Y = l^\infty = \{x = (x_1, x_2, \ldots, x_i, \ldots) : \sup_{i\geq 1} |x_i| < \infty\}$, $C_4 = \{x = (x_1, x_2, \ldots, x_i, \ldots) \in l^\infty : x_i \geq 0, i = 1, 2, \ldots\}$. Then, $\operatorname{int} C_4 \neq \emptyset$.

(2) There exist cones with empty interiors as given in following example.

Example 2.2

(i) Let $Y = L^p(\Omega)$, where $\Omega \subset R^n, 0 < mes\Omega < \infty, 1 \leq p < \infty$, let $C_5 = \{x \in L^p(\Omega) : x(t) \geq 0, \forall \text{ a.e. } t \in \Omega\}$. Then, $\operatorname{int} C_5 = \emptyset$.
(ii) Let $Y = l^p, 1 \leq p < \infty$, $C_6 = \{x = (x_1, x_2, \ldots, x_i, \ldots) \in l^p : x_i \geq 0, i = 1, 2, \ldots\}$. Then, $\operatorname{int} C_6 = \emptyset$.

(3) In [16], Qiu obtained some criteria for checking whether or not convex cones are having nonempty interiors in various kinds of locally convex spaces.

Throughout this paper, unless otherwise specified, we always assume $e \in \operatorname{int} C$ is a fixed element in Y and $B^* = \{l \in C^* : l(e) = 1\}$.

Lemma 2.2 *Let C be a closed convex pointed cone with* $\operatorname{int} C \neq \emptyset$. *Then,*

$$C = \{y \in Y : l(y) \geq 0, \quad \forall l \in B^*\},$$

and

$$\operatorname{int} C = \{y \in Y : l(y) > 0, \quad \forall l \in B^*\}.$$

Proof Since C is a closed convex pointed cone with $\operatorname{int} C \neq \emptyset$, by Lemma 3.21 in [8], we have $C = \{y \in Y : l(y) \geq 0, \forall l \in C^*\}$ and $\operatorname{int} C = \{y \in Y : l(y) > 0, \forall l \in C^* \backslash \{0\}\}$. Further, since B^* is a w^*-compact base of C^* by Lemma 2.1, we get

$$C = \{y \in Y : l(y) \geq 0, \forall l \in B^*\} \quad \text{and} \quad \operatorname{int} C = \{y \in Y : l(y) > 0, \forall l \in B^*\}.$$

The following real-valued function plays an important role in many areas of vector optimization problems. □

Definition 2.2 [14] For fixed $e \in \operatorname{int} C$ and $q \in Y$, Tammer function $h_e(\cdot, q) : Y \to R$ is defined by:

$$h_e(y, q) = \inf\{t \in R : y \in te + q - C\}, y \in Y.$$

Tammer function $h_e(\cdot, q)$ has the following salient properties.

Lemma 2.3 [14,19] *For fixed $e \in \operatorname{int} C$ and any $q \in Y$, we have*

(i) $h_e(y, q) < t \Leftrightarrow y \in te + q - \operatorname{int} C$;
(ii) $h_e(y, q) \leq t \Leftrightarrow y \in te + q - C$; *and*
(iii) $h_e(\cdot, q)$ *is a continuous convex function on Y and strictly monotone.*

Following the idea of Proposition 2.2 in [2], we obtain the following result.

Lemma 2.4 *Let $B^* = \{l \in C^* : l(e) = 1\}$, where $e \in \operatorname{int} C$. Then, for any $q \in Y$,*

$$h_e(y, q) = \max\{l(y) - l(q) : l \in B^*\} = l_0(y - q), y \in Y,$$

where $l_0 = l_0(y, q) \in B^*$.

Proof Since $h_e(y, q) = \inf\{t \in R : y \in te + q - C\}$ and C is closed, $y \in h_e(y, q)e + q - C$. So, by Lemma 2.2, we obtain

$$l(h_e(y, q)e + q - y) \geq 0, \quad \forall l \in B^*.$$

This implies that $h_e(y, q) \geq l(y - q), \forall l \in B^*$, resulting in

$$h_e(y, q) \geq \sup\{l(y) - l(q) : l \in B^*\}. \tag{1}$$

Conversely, let $t_0 = \sup\{l(y) - l(q) : l \in B^*\}$. Then

$$l(y) - l(q) \leq t_0 = t_0 l(e), \quad \forall l \in B^*.$$

Note that l is linear, so $l(y - q - t_0 e) \leq 0, \forall l \in B^*$. By Lemma 2.2 again, we have $y \in t_0 e + q - C$. From the definition of h_e, we get

$$t_0 \geq h_e(y, q) = \inf\{t \in R : y \in te + q - C\}.$$

This together with (1) yields

$$h_e(y, q) = \sup\{l(y) - l(q) : l \in B^*\}.$$

Further, since B^* is a w^*-compact set and for any fixed $y, q, l(y - q) : Y^* \to R$ is w^*-continuous, there exists $l_0 = l_0(y, q) \in B^*$ such that

$$\sup\{l(y) - l(q) : l \in B^*\} = \max\{l(y) - l(q) : l \in B^*\} = l_0(y - q).$$

The proof is complete. □

Remark 2.3 Let $Y = R^n$, $C = \{x = (x_1, x_2, \ldots, x_n) \in R^n : x_i \geq 0, i = 1, 2, \ldots, n\}$, and $e = (e_1, e_2, \ldots, e_n) \in \operatorname{int} C$ and $q = (q_1, q_2, \ldots, q_n) \in Y$. Then, the function $h_e(\cdot, q)$ may be written as

$$h_e(y, q) = \max\left\{\frac{y_i - q_i}{e_i} : 1 \leq i \leq n\right\}, \quad \forall y = (y_1, y_2, \ldots, y_n).$$

Lemma 2.5 [12] *For fixed $e \in \operatorname{int} C$, $h_e(\cdot, \cdot)$ is continuous on $Y \times Y$. Next, we recall some well known results and concepts concerning set-valued functions.*

Let $A \subseteq X$ be a nonempty subset and $F : A \longrightarrow 2^Y$ a set-valued function with nonempty values (i.e., $\forall x \in A$, $F(x) \neq \emptyset$). F is called upper semicontinuous at $x_0 \in A$ if for every neighborhood V containing $F(x_0)$, there is a neighborhood U of x_0 such that

$$F(x) \subseteq V, \quad \forall x \in U.$$

F is upper semicontinuous if F is upper semicontinuous at every point $x \in A$.

Definition 2.3 [14] Let C be a closed convex cone. F is called C-upper semicontinuous at $x_0 \in A$ if for every neighborhood V containing $F(x_0)$, there is a neighborhood U of x_0 such that

$$F(x) \subseteq V + C, \quad \forall x \in U.$$

Definition 2.4 [15] Let $A \subseteq X$ be a nonempty convex subset. $F : A \longrightarrow 2^Y$ is said to be quasiconvex if for every $a \in Y$, the level set of F at a:

$$lev_F(a) = \{x \in A : \text{ there is } y \in F(x) \text{ such that } y \leq a\}$$

is convex.

Remark 2.4

(i) If F is single-valued and Y is a topological lattice, in which the lattice order is generated by a closed pointed convex cone. For any $y_1, y_2 \in Y$, let $\sup\{y_1, y_2\}$ denote the supremum of y_1, y_2. A single valued function $F : A \to Y$ is said to be quasiconvex, if for any $x_1, x_2 \in A$, $t \in [0, 1]$, we have

$$F(tx_1 + (1 - t)x_2) \in \sup\{F(x_1), F(x_2)\} - C.$$

(ii) Suppose, for a special case, that $Y = R^n$, and $C = R^n_+$. A single valued function $F = (f_1, f_2, \ldots, f_n) : A \to Y$ is quasiconvex if and only if every f_i, $i = 1, 2, \ldots, n$, is quasiconvex in the usual sense, i.e., for any $x_1, x_2 \in A$, $t \in [0, 1]$,

$$f_i(tx_1 + (1 - t)x_2) \leq \max\{f_i(x_1), f_i(x_2)\}.$$

Example 2.3 Let $Y = l^\infty = \{x = (x_1, x_2, \ldots, x_i, \ldots) : \sup_{i \geq 1} |x_i| < \infty\}$, $C = \{x = (x_1, x_2, \ldots, x_i, \ldots) \in l^\infty : x_i \geq 0, i = 1, 2, \ldots\}$ and $A = R^+$. $F : A \to 2^Y$ is defined by

$$F(x) = \left\{\left(x, \frac{x}{2}, \ldots, \frac{x}{n}, \ldots\right), \left(-x, \frac{-x}{2}, \ldots, \frac{-x}{n}, \ldots\right)\right\}, \quad \forall x \in A.$$

Then, F is a quasiconvex set-valued function.

In fact, we only need to check, for any $a = (a_1, a_2, \ldots, a_i, \ldots) \in Y$, if

$$\mathrm{lev}_F(a) = \{x \in A : \text{ there exists } y \in F(x) \text{ such that } y \leq a\}$$

is convex. Without loss of generality, we assume that $\mathrm{lev}_F(a) \neq \emptyset$, let x_1, x_2 be any two points of the above level set and $\lambda \in (0, 1)$. Noting that $x_1, x_2 \geq 0$, by the definition of F, we assume that $\frac{-x_1}{n} \leq a_n, \frac{-x_2}{n} \leq a_n, \quad n = 1, 2, \ldots$, then we have

$$-\frac{\lambda x_1 + (1-\lambda)x_2}{n} = \lambda\left(\frac{-x_1}{n}\right) + (1-\lambda)\left(\frac{-x_2}{n}\right) \leq \lambda a_n + (1-\lambda)a_n = a_n, \quad n = 1, 2, \ldots.$$

This shows that $\lambda x_1 + (1-\lambda)x_2 \in \mathrm{lev}_F(a)$. It is completed the proof. □

Remark 2.5 In general, suppose that F is quasiconvex, and that ϕ is monotone and convex. Then, it is not necessary for ϕF to be quasiconvex as shown in [14]. But, we have the following useful result.

Lemma 2.6 *Let $A \subseteq X$ be a nonempty convex subset and F be quasiconvex set-valued map. Then for any $q \in Y$, $h_e F$ is also quasiconvex.*

Proof For any $q \in Y$, we have to check, for any $t_0 \in R$, if

$$lev_{hF}(t_0) = \{x \in A : \text{ there exists } t \in h_e(F(x), q) \quad \text{such that} \quad t \leq t_0\}$$

is convex. Without loss of generality, we assume that $lev_{hF}(t_0) \neq \emptyset$, let x_1, x_2 be any two points of the above level set and $\lambda \in (0, 1)$. Then there exist $y_i \in F(x_i)$ with $t_i = h_e(y_i, q) \leq t_0$, $i = 1, 2$. Since C is closed,

$$y_i \in q + t_i e - C, \quad i = 1, 2.$$

Noticing that $t_i = h_e(y_i, q) \leq t_0$, by Lemma 2.3, we have $y_i \leq q + t_0 e$, $i = 1, 2$. By the quasiconvexity of F, there is a $y_\lambda \in F(\lambda x_1 + (1-\lambda)x_2)$ such that $y_\lambda \leq q + t_0 e$, i.e., $y_\lambda \in q + t_0 e - C$. This implies that $h_e(y_\lambda, q) \leq t_0$. Therefore, $\lambda x_1 + (1-\lambda)x_2 \in \mathrm{lev}_{hF}(t_0)$. The proof is complete. □

Remark 2.6 Lemma 2.6 is different from Proposition 2.3 in [15] in the following aspects:

(i) The condition that the set-valued function is compact valued is removed; and
(ii) the order in Y is given by C.

Remark 2.7 It is worthwhile noticing that the property (ii) of Lemma 2.3 plays an important role.

Now, we consider the following vector optimization problem with set-valued functions:

$$\text{(VP)} \quad \min_{x \in A} F(x).$$

Definition 2.5 Let $e \in \mathrm{int}\, C$ be fixed element, $\varepsilon \geq 0$, $x_\varepsilon \in A$ is called an εe-weak efficient solution for (VP), written as $x_\varepsilon \in \mathrm{WAE}(F, A, C, \varepsilon e)$, if there exists $y_\varepsilon \in F(x_\varepsilon)$ such that

$$F(A) \cap (y_\varepsilon - \varepsilon e - \mathrm{int}\, C) = \emptyset.$$

Remark 2.8

(i) If $\varepsilon = 0$, the definition of εe-weak efficient solution coincides with the definition of weak efficient solution. In the sequel, the symbol $\mathrm{WE}(F, A, C)$ denotes the set of weak efficient solutions for (VP).
(ii) When the set-valued function degenerates to vector-valued function in (VP), we have

$$Y = R^n, \quad C = R^n_+, e = (1, 1, \ldots, 1) \in \mathrm{int}\, C.$$

In this case, the definition of ε-weak minimum of (VP) in [3,4] coincides with the definition of εe-weak efficient solution in this paper.

3 Characterization in terms of the scalarization

Now, consider the following set-valued scalar minimization problem induced by (VP).

$$(P_{\mathrm{q}}) \quad \min_{x \in A} h_e(F(x), q).$$

Definition 3.1 A point $x_\varepsilon \in A$ is called an ε-approximate optimal solution of (P_{q}), written as $x_\varepsilon \in S_{\mathrm{q}}(\varepsilon)$, if there exists a $y_\varepsilon \in F(x_\varepsilon)$ such that

$$h_e(y_\varepsilon, \mathrm{q}) - \varepsilon \leq h_e(y, q), \quad \forall x \in A,\ y \in F(x).$$

Proposition 3.1 *For any $q \in Y$, we have*

$$S_{\mathrm{q}}(\varepsilon) \subseteq \mathrm{WAE}(F, A, C, \varepsilon e).$$

Proof Assume that there exists an $\overline{x} \in S_{\mathrm{q}}(\varepsilon)$, but $\overline{x} \notin \mathrm{WAE}(F, A, C, \varepsilon e)$. Then for any $y \in F(\overline{x})$ there exist $x_y \in A$ and $y' \in F(x_y)$ such that $y' \in y - \varepsilon e - \mathrm{int}\, C$. Hence, by Lemma 2.2, we have $l(y' - y + \varepsilon e) < 0, \forall l \in B^*$. This implies

$$l(y' - q) + \varepsilon < l(y - q), \quad \forall l \in B^*. \tag{2}$$

On the other hand, by Lemma 2.4, there exists a $l_0 \in B^*$ such that $h_e(y', q) = l_0(y' - q)$. Thus, by (2), we obtain

$$\begin{aligned} h_e(y', q) = l_0(y' - q) &< l_0(y - q) - \varepsilon \\ &\leq \max\{l(y - q) : l \in B^*\} - \varepsilon \\ &= h_e(y, q) - \varepsilon, \end{aligned}$$

i.e., for any $y \in F(\overline{x})$, there exists a $y' \in F(A)$ such that

$$h_e(y, q) - \varepsilon > h_e(y', q),$$

which contradicts $\overline{x} \in S_q(\varepsilon)$. The proof is complete. □

Proposition 3.2 *Assume that $\overline{x} \in \mathrm{WAE}(F, A, C, \varepsilon e)$. Then, there exists a $q \in Y$ such that $\overline{x} \in S_q(\varepsilon)$.*

Proof Since $\overline{x} \in \mathrm{WAE}(F, A, C, \varepsilon e)$, there exists a $\overline{y} \in F(\overline{x})$ such that

$$F(A) \cap (\overline{y} - \varepsilon e - \mathrm{int}\, C) = \emptyset. \tag{3}$$

By Lemma 2.2, for any $l \in B^*$, we have $l(-\mathrm{int}\, C) < 0$. So, by (3), for any $y \in F(A)$, we obtain

$$l(y - \overline{y} + \varepsilon e) \geq 0, \quad \forall l \in B^*.$$

This implies

$$l(y - \overline{y}) \geq -\varepsilon, \quad \forall l \in B^*.$$

By Lemma 2.4, we have

$$h_e(y, \overline{y}) = \max\{l(y - \overline{y}) : l \in B^*\} \geq -\varepsilon, \quad \forall y \in F(A).$$

Note that $h_e(\overline{y}, \overline{y}) = \max\{l(\overline{y} - \overline{y}) : l \in B^*\} = 0$. Thus, we have

$$h_e(\overline{y}, \overline{y}) - \varepsilon \leq h_e(y, \overline{y}), \quad \forall y \in F(A).$$

That is, $\overline{x} \in S_{\overline{y}}(\varepsilon)$. The proof is complete. □

Utilizing Propositions 3.1 and 3.2, we immediately obtain the following theorem.

Theorem 3.1 *Let* $\varepsilon \geq 0$*, we have*

$$\mathrm{WAE}(F, A, C, \varepsilon e) = \bigcup\{S_q(\varepsilon) : q \in Y\}.$$

Remark 3.1 Theorem 3.1 is different from Theorem 2.1 in [17]. We derive the scalar characterization without imposing any convexity assumption on objective functions and feasible set.

Below, we give the existence result of εe-weak efficient solution for (VP).

A subset $M \subseteq Y$ is said to be C-bounded if there exists a $y_0 \in Y$ such that $M \subseteq y_0 + C$.

Theorem 3.2 *Let* $F(A)$ *be* C*-bounded and* $\varepsilon > 0$*. Then* $\mathrm{WAE}(F, A, C, \varepsilon e) \neq \emptyset$.

Proof Since $F(A)$ be C-bounded, there exists a $y_0 \in Y$ such that

$$y \geq y_0, \quad \forall y \in F(A).$$

On the other hand, by Lemma 2.3 and Remark 2.1, for $q \in Y$, $h_e(\cdot, q)$ is monotone, so we have

$$h_e(y, q) \geq h_e(y_0, q), \quad \forall y \in F(A).$$

Therefore, $\inf\{h_e(y, q) : y \in F(A)\}$ exists. By the definition of infimum, for $\varepsilon > 0$, there exists a $\overline{y} \in F(A)$ such that

$$h_e(\overline{y}, q) < \inf\{h_e(y, q) : y \in F(A)\} + \varepsilon.$$

This shows that

$$h_e(\overline{y}, q) - \varepsilon \leq h_e(y, q), \quad \forall y \in F(A). \tag{4}$$

Since $\overline{y} \in F(A)$, there exists an $\overline{x} \in A$ with $\overline{y} \in F(\overline{x})$ and $\overline{y}$ satisfying (4). That is, $\overline{x} \in S_q(\varepsilon)$. By Theorem 3.1, we have $\overline{x} \in \mathrm{WAE}(F, A, C, \varepsilon e)$. The proof is complete. □

Remark 3.2 From Theorem 3.2, we see that the existence conditions for approximate solutions is very weak.

4 Relations between approximate solutions and weak efficient solutions

Proposition 4.1 *If* $0 \leq \varepsilon_1 \leq \varepsilon_2$*, then* $\mathrm{WAE}(F, A, C, \varepsilon_1 e) \subseteq \mathrm{WAE}(F, A, C, \varepsilon_2 e)$.

Proof For any $x_1 \in \mathrm{WAE}(F, A, C, \varepsilon_1 e)$, by Theorem 3.1, there exists $q \in Y$ with $x_1 \in S_q(\varepsilon_1)$. Namely, there exists a $y_1 \in F(x_1)$ such that

$$h_e(y_1, q) - \varepsilon_1 \leq h_e(y, q), \quad \forall y \in F(A). \tag{5}$$

Note that $\varepsilon_1 \leq \varepsilon_2$, it is clear from (5) that

$$h_e(y_1, q) - \varepsilon_2 \leq h_e(y_1, q) - \varepsilon_1 \leq h_e(y, q), \quad \forall y \in F(A).$$

This means that $x_1 \in S_q(\varepsilon_2)$. By Theorem 3.1 again, we have $x_1 \in \mathrm{WAE}(F, A, C, \varepsilon_2 e)$. The proof is complete. □

Theorem 4.1 *Let $e \in \operatorname{int} C$. Then,*

$$\bigcap_{\varepsilon>0} \mathrm{WAE}(F, A, C, \varepsilon e) = \mathrm{WE}(F, A, C).$$

Proof Firstly, we prove $\mathrm{WE}(F, A, C) \subseteq \bigcap_{\varepsilon>0} \mathrm{WAE}(F, A, C, \varepsilon e)$.
For any $z \in \mathrm{WE}(F, A, C)$, there exists a $y_z \in F(z)$ such that

$$F(A) \cap (y_z - \operatorname{int} C) = \emptyset. \tag{6}$$

If there exists an $\varepsilon_0 > 0$ such that $z \notin \mathrm{WAE}(F, A, C, \varepsilon_0 e)$, then for any $y \in F(z)$, there exists $y' \in F(A)$ such that $y' - y + \varepsilon_0 e \in -\operatorname{int} C$. This implies that $y' \in y - \varepsilon_0 e - \operatorname{int} C \subseteq y - \operatorname{int} C$, which contradicts (6). Since $z \in \mathrm{WE}(F, A, C)$ is arbitrary, we have $\mathrm{WE}(F, A, C) \subseteq \bigcap_{\varepsilon>0} \mathrm{WAE}(F, A, C, \varepsilon e)$.

Now, we prove $\bigcap_{\varepsilon>0} \mathrm{WAE}(F, A, C, \varepsilon e) \subseteq \mathrm{WE}(F, A, C)$.
Suppose that it is false. Then, there exist an $\overline{x} \in \bigcap_{\varepsilon>0} \mathrm{WAE}(F, A, C, \varepsilon e)$, but $\overline{x} \notin \mathrm{WE}(F, A, C)$. Then, for any $y \in F(\overline{x})$, there exist $x_y \in A$ and $y' \in F(x_y)$ such that $y' \in y - \operatorname{int} C$. Thus, there exists $d \in \operatorname{int} C$ such that

$$-d = y' - y \in F(A) - y. \tag{7}$$

Since $d \in \operatorname{int} C$, there exists an $\varepsilon_0 > 0$ such that

$$d - \varepsilon_0 e \in \operatorname{int} C. \tag{8}$$

By (7), we have $-d + \varepsilon_0 e \in F(A) - y + \varepsilon_0 e$. This together with (8) yields

$$(F(A) - y + \varepsilon_0 e) \cap (-\operatorname{int} C) \neq \emptyset.$$

This shows that $\overline{x} \notin \mathrm{WAE}(F, A, C, \varepsilon_0 e)$, which contradicts $\overline{x} \in \bigcap_{\varepsilon>0} \mathrm{WAE}(F, A, C, \varepsilon e)$. The proof is complete. □

Remark 4.1 Theorem 4.1 generalizes and improves the corresponding result of Dutta and Vetrivel (see [4, Proposition 2.1]). In particular, in several directions as indicated below.

(1) The setting of R^n is generalized to locally convex space.
(2) The condition that the objective function is convex is removed;
(3) the vector-valued function is extended to set-valued function.

Theorem 4.2 *Let $\varepsilon \geq 0$, let $(\varepsilon_n)_{n\in\Lambda}$ be positive numbers converging to ε, let A be closed and let $F : A \longrightarrow 2^Y$ be a C-upper semicontinuous set-valued function. Then,*

$$\limsup \mathrm{WAE}(F, A, C, \varepsilon_n e) \subseteq \mathrm{WAE}(F, A, C, \varepsilon e),$$

where $\limsup \mathrm{WAE}(F, A, C, \varepsilon_n e) = \{x \in A :$ *there exists a subnet* $\{x_m\}$ *such that* $x_m \in \mathrm{WAE}(F, A, C, \varepsilon_m e)$ *and* $x_m \to x\}$.

Proof Assume that there exists an $\overline{x} \in \limsup \mathrm{WAE}(F, A, C, \varepsilon_n e)$, but $\overline{x} \notin \mathrm{WAE}(F, A, C, \varepsilon e)$. Then, for any $y \in F(\overline{x})$ there exist $x_y \in A$ and $y' \in F(x_y)$ such that $y' - y + \varepsilon e \in -\operatorname{int} C$. Namely, $y \in y' + \varepsilon e + \operatorname{int} C$. $y \in F(\overline{x})$ is arbitrary, we have

$$F(\overline{x}) \subseteq F(A) + \varepsilon e + \operatorname{int} C.$$

Since $F(A) + \varepsilon e + \operatorname{int} C$ is open and F is C-upper semicontinuous, there exists a neighborhood $N(\overline{x})$ of $\overline{x}$ such that

$$F(N(\overline{x})) \subseteq F(A) + \varepsilon e + \operatorname{int} C + C \subseteq F(A) + \varepsilon e + \operatorname{int} C. \tag{9}$$

Note that $\overline{x} \in \limsup \mathrm{WAE}(F, A, C, \varepsilon_n e)$. Thus, there exists a subnet $\{x_m\}_{m\in\Lambda'\subseteq\Lambda}$ such that $x_m \in \mathrm{WAE}(F, A, C, \varepsilon_m e)$ and $x_m \to \overline{x}$. So, there is a $\tau_0 \in \Lambda'$ such that

$$x_m \in N(\overline{x}), \quad m \geq \tau_0.$$

This together with (9) yields

$$F(x_m) \subseteq F(A) + \varepsilon e + \operatorname{int} C.$$

Then, for any $z \in F(x_m)$, there exist $y' \in F(A)$, $d \in \operatorname{int} C$ such that

$$z = y' + \varepsilon e + d. \tag{10}$$

Observe that $d \in \operatorname{int} C$. Then, there is a neighborhood U of zero such that

$$d + U \subseteq \operatorname{int} C.$$

Since $\varepsilon_n \to \varepsilon$, $(\varepsilon - \varepsilon_m)e \to 0$. Thus, there exists a $\tau_1 \in \Lambda$ such that

$$(\varepsilon - \varepsilon_m)e \in U, \quad m \geq \tau_1.$$

and hence

$$d + (\varepsilon - \varepsilon_m)e \in \operatorname{int} C, \quad m \geq \tau_1. \tag{11}$$

Therefore, for any $z \in F(x_m)$, by (10) and (11), we have

$$\begin{aligned} z &= y' + \varepsilon e + d \\ &= y' + \varepsilon_m e + (\varepsilon - \varepsilon_m)e + d \\ &\subseteq F(A) + \varepsilon_m e + \operatorname{int} C, \quad m \geq \tau_1, m \geq \tau_0. \end{aligned}$$

This shows that $(z - \varepsilon_m e - \operatorname{int} C) \cap F(A) \neq \emptyset$, $\forall z \in F(x_m)$, which contradicts $x_m \in \mathrm{WAE}(F, A, C, \varepsilon_m e)$. The proof is complete. □

Remark 4.2 Theorem 4.2 generalizes and improves the corresponding result of Deng (see [3, Proposition 1]). In particular, in several directions as indicated below.

(1) The setting of R^n is generalized to locally convex space.
(2) The condition that the objective function is convex is removed.
(3) The vector-valued function is extended to set-valued function.

5 Connectedness of approximate solution set

Lemma 5.1 [7] *Let $M \subseteq X$ be a nonempty connected set and let $H : M \longrightarrow 2^Y$ be an upper semicontinous set-valued function with nonempty connected value. Then, $H(M)$ is connected.*

Lemma 5.2 *Assume that $A \subseteq X$ is a nonempty convex subset and that $F : A \longrightarrow 2^Y$ is a quasiconvex set-valued function. Then, for any $q \in Y$ and $\varepsilon \geq 0$, $S_q(\varepsilon)$ is convex.*

Proof Let $x_i \in S_q(\varepsilon)$, $i = 1, 2$, $t \in (0, 1)$. Then, there exists a $y_i \in F(x_i)$ such that

$$h_e(y_i, q) - \varepsilon \leq h_e(y, q), \quad \forall y \in F(A). \tag{12}$$

Denoting $r_q = \inf\{h_e(y, q) : y \in F(A)\}$, by (12), we have

$$h_e(y_i, q) \leq r_q + \varepsilon, \quad i = 1, 2. \tag{13}$$

Since F is quasiconvex, it follow from Lemma 2.6 that for $q \in Y$, $h_e F$ is also quasiconvex. Hence, the level set

$$\mathrm{lev}_{h_e F}(r_q + \varepsilon) = \{x \in A : \text{ there exists } t \in h_e(F(x), q) \quad \text{such that} \quad t \leq r_q + \varepsilon\}$$

is convex. By (13), we have $x_i \in lev_{hF}(r_q+\varepsilon)$. Consequently, $\lambda x_1+(1-\lambda)x_2 \in lev_{h_e F}(r_q+\varepsilon)$. This implies that there exists a $\overline{y} \in F(\lambda x_1 + (1-\lambda)x_2)$ such that

$$h_e(\overline{y}, q) \leq r_q + \varepsilon,$$

i.e.,

$$h_e(\overline{y}, q) - \varepsilon \leq r_q \leq h_e(y, q), \quad \forall y \in F(A).$$

This together with $\overline{y} \in F(\lambda x_1 + (1-\lambda)x_2)$ yields $\lambda x_1 + (1-\lambda)x_2 \in S_q(\varepsilon)$. The proof is complete. □

For $\varepsilon \geq 0$, define the following set-valued function:

$$G : Y \to 2^A, \quad \text{where } G(q) = S_q(\varepsilon),\ q \in Y.$$

Lemma 5.3 *Let $A \subseteq X$ be a nonempty compact subset and let $F : A \longrightarrow 2^Y$ be an upper semicontinuous set-valued function with compact value. Then, G is upper semicontinuous on Y.*

Proof Assuming that G is not upper semicontinuous on Y. Then, there exist $q_0 \in Y$ and a neighborhood $\mathcal{N}(G(q_0))$ of $G(q_0)$ such that for any $V \in \mathcal{N}(0)$ (where $\mathcal{N}(0)$ denote the neighborhood base of zero in Y), there exists a $q_V \in q_0 + V$ with

$$G(q_V) \nsubseteq \mathcal{N}(G(q_0)). \tag{14}$$

It is clear that $q_V \to q_0$. By (14) there exists an $x_V \in G(q_V)$ such that $x_V \notin \mathcal{N}(G(q_0))$. Since $x_V \in G(q_V) = S_{q_V}(\varepsilon)$, there is a $y_V \in F(x_V)$ such that

$$h_e(y_V, q_V) - \varepsilon \leq h_e(y, q_V), \quad \forall y \in F(A) \tag{15}$$

Since A is a nonempty compact subset and $F : A \longrightarrow 2^Y$ is an upper semicontinuous set-valued function with compact value, it is clear from Proposition 11 in [1], then $F(A)$ is compact. Without loss of generality, we can assume that $y_V \to \overline{y} \in F(A)$, $x_V \to \overline{x} \in A$. Moreover, by Proposition 7 in [1], F is closed, hence $\overline{y} \in F(\overline{x})$. Further, by Lemma 2.5, $h_e(\cdot,\cdot)$ is continuous on $Y \times Y$, taking limit in (15), we have

$$h_e(\overline{y}, q_0) - \varepsilon \leq h_e(y, q_0), \quad \forall y \in F(A).$$

This together with $\overline{y} \in F(\overline{x})$ yields $\overline{x} \in S_{q_0}(\varepsilon) = G(q_0)$ which contradicts $x_V \notin \mathcal{N}(G(q_0))$. This completes the proof.

□

Theorem 5.1 *Assume that $A \subseteq X$ is a nonempty compact convex subset and that $F : A \longrightarrow 2^Y$ is a upper semicontinuous quasiconvex set-valued function with compact value. Then, for $\varepsilon \geq 0$, WAE$(F, A, C, \varepsilon e)$ is connected.*

Proof For $\varepsilon \geq 0$ and any $q \in Y$, since $A \subseteq X$ is a convex subset and $F : A \longrightarrow 2^Y$ is a quasiconvex set-valued function, it is clear from Lemma 5.2 that $S_q(\varepsilon)$ is convex. Hence,

$S_q(\varepsilon)$ is connected. Moreover, it follows from Lemma 5.3 that G is upper semicontinuous on Y. Thus,

$$\bigcup\{G(q) : q \in Y\} = \bigcup\{S_q(\varepsilon) : q \in Y\}$$

is connected by Lemma 5.1. On the other hand, by Theorem 3.1, we have $\mathrm{WAE}(F, A, C, \varepsilon e) = \bigcup\{S_q(\varepsilon) : q \in Y\}$. Consequently, $\mathrm{WAE}(F, A, C, \varepsilon e)$ is connected. The proof is complete. $\square$

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